Exercise Sheet #9

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P1. **(Area interpretation of the integral)** Let $(X, \mathcal{B}, \mu)$ a s-finite probability space and $f : X \to [0, \infty]$ a measurable function. Let $R_f = \{(x, t) \in X \times \mathbb{R} : 0 \le t \le f(x)\}$. Prove that

(a) If f is measurable then $R_f \in \mathcal{B} \otimes \text{Borel}(\mathbb{R})$ and

$$\int_X f d\mu = (\mu \overset{\text{c-s}}{\otimes} \lambda)(R_f).$$

where λ is the Lebesgue measure.

(b) If $f : X \to \mathbb{R}$ integrable then

$$\int_X f d\mu = (\mu \overset{\text{c-s}}{\otimes} \lambda)(R_{f^+}) - (\mu \overset{\text{c-s}}{\otimes} \lambda)(R_{f^-}).$$

P2. Denote λ the Lebesgue measure on $\mathbb{R}$. Show that if $A, B \subseteq$ are Lebesgue measurable sets such that $\lambda(A), \lambda(B) > 0$ then there is $t \in \mathbb{R}$ satisfying $\lambda(A \cap (B - t)) > 0$.

P3. Consider $X = [0, 1]$ equipped with the standard topology and $Y = [0, 1]$ equipped with the discrete topology. Define $\varphi : C_c(X \times Y) \to \mathbb{C}$ by $\varphi(f) = \sum_y \int_0^1 f(x, y) dx$ where the integral corresponds to the Riemman integral. Let μ be the Radon measure corresponding to φ. Is μ a product of $\lambda|_{\text{Borel}([0,1])}$ and counting measure ?
Hint: Consider the rectangle $\{0\} \times [0, 1]$.